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# The Four-Loop QCD $\beta$ -Function and Anomalous Dimensions

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## Abstract

The four-loop  $\beta$ -function of quantum chromodynamics is calculated and agreement is found with the previous result. The anomalous dimensions of the quark-gluon vertex, and quark, gluon and ghost fields are given for a general compact simple Lie group.

## 1 Introduction

The renormalization group properties of Quantum Chromodynamics were the reason of acceptance of this theory as the theory of strong interactions. The central rôle played by the QCD  $\beta$ -function, calculated at the one- [1], two- [2], three- [3] and finally at the four-loop [4] level, cannot be overestimated in this respect.

The calculation of the last known, four-loop, term in the expansion of the  $\beta$ -function, was performed by only one group [4]. The authors evaluated "...of the order of 50.000 4-loop diagrams". These two facts lead to the conclusion, stressed many times (in *e.g.* [5]) that it is not only justified, but also necessary to independently evaluate this quantity. The present work is intended to fulfill this need.

Apart from the  $\beta$ -function itself, we present two new results. The first one is the complete set of  $\overline{\text{MS}}$  anomalous dimensions at the four-loop level in the linear gauge and with color structures of a general compact simple Lie group. These give, in a compact form, the complete set of four-loop QCD renormalization constants. We notice that the case of the SU(N) group has been solved in [5] with the assumption, however, that the

four-loop  $\beta$ -function is correct. Our second new result is of a more technical nature, but should simplify future renormalization group calculations at this level of the perturbative expansion. To this end, we derived the necessary set of divergent parts of the four-loop fully massive tadpole master integrals. A purely numeric result was available from [6], but could not be used for our purpose.

This paper is organized as follows. The next section presents the methods used in the calculation, as well as some efficiency considerations. Subsequently, our results for the anomalous dimensions are listed. Conclusions and remarks close the main part of the paper. The expansions of the tadpole master integrals are contained in the Appendix.

## 2 Calculation

The calculation of renormalization group parameters, *i.e.* anomalous dimensions and beta functions allows for vast simplifications in comparison to the actual evaluation of Green functions with kinematic invariants in the physical region. Dimensional regularization and the  $\overline{\text{MS}}$  scheme are particularly well suited for this kind of problems, since they make it possible to manipulate dimensionful parameters of the theory. In fact, ever since the introduction of the Infrared Rearrangement [7], it is known how to set most of them to zero and avoid spurious infrared poles. With the advent of the  $R^*$  operation [8], infrared divergences are even allowed in individual diagrams and only compensated by counterterms afterward.

At the four-loop level two techniques seem to be most promising. One is a global  $R^*$  operation [5], where one would set all of the external momenta to zero and also almost all of the masses, keeping just one massive line. The spurious infrared divergences are then compensated by adding a global counterterm. The advantage of this approach is that the whole problem can be reduced to the calculation of three-loop massless propagators, for which there exists a well tested and efficient FORM [9] package, MINCER [10]. The disadvantage is that the construction of the global counterterm is not trivial. In fact, up to now it has not been possible for the gluon propagator.

A second technique consists in setting all of the external momenta to zero, but keeping a common non-zero mass for all the internal lines [11, 4, 12]. The advantage of this approach is that one never encounters any infrared divergences. One minor part of the price for this convenience is the necessity of a gluon mass counterterm. The more problematic part is, of course, the calculation of the divergent parts of four-loop tadpole diagrams that occur. One way to do this is to generalize the algorithms of [12] to the four-loop level. We, however, decided to use integration-by-parts identities to reduce all of the integrals to a set of master integrals depicted in Fig. 1. The divergent parts of the latter were then calculated as described in Appendix A.

Instead of developing a dedicated software for the reduction of tadpole integrals<sup>1</sup> we used our own implementation of the Laporta algorithm [14] in the form of the C++ library

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<sup>1</sup>Such a software has been developed in [13].

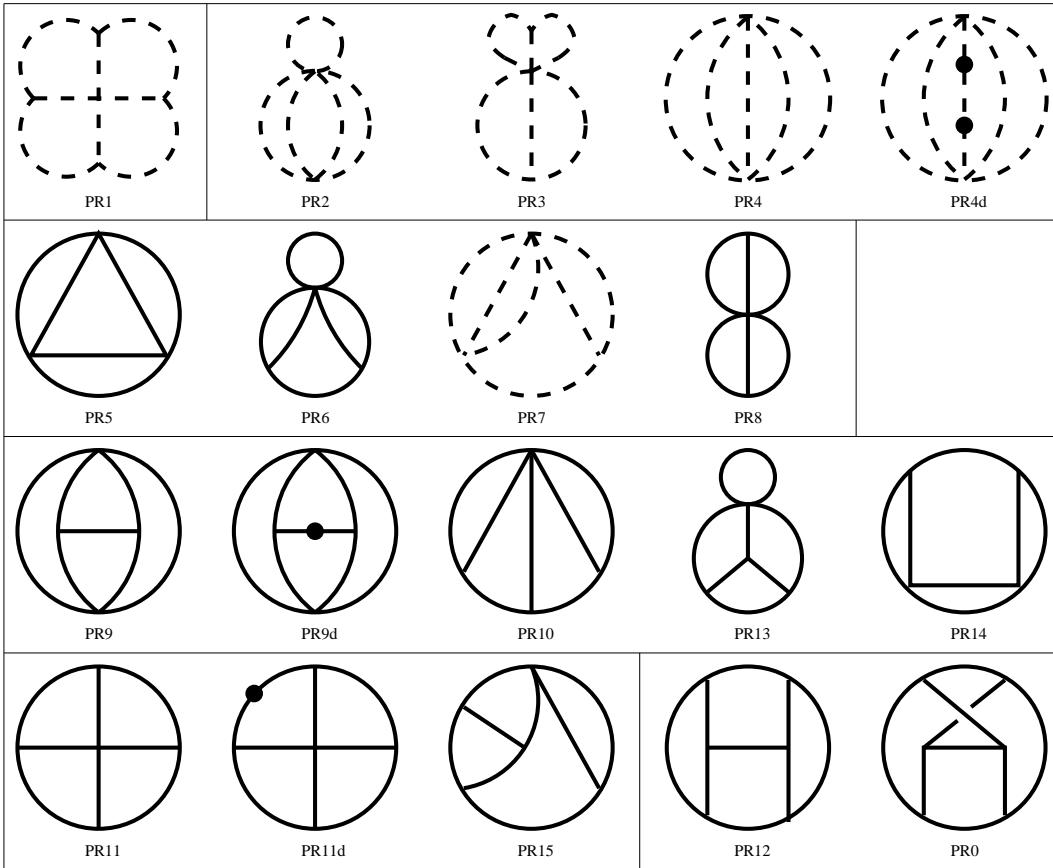


Figure 1: Completely massive four-loop tadpole master integrals. Dashed lines mark those integrals which are needed to one order higher in the  $\epsilon$  expansion, *i.e.* up to finite parts for this calculation.

Prototype	number of lines	number of integrals	denominator powers	numerator powers
PR1	4	6764	7	3
PR2		8575	6	3
PR3	5	19402	7	4
PR4		2659	5	2
PR5		8944	5	3
PR6	6	26614	7	5
PR7		15058	6	4
PR8		21528	7	4
PR9		15906	5	4
PR10	7	28244	7	5
PR13		6988	6	4
PR14		16157	7	5
PR11	8	11654	5	5
PR15		10973	6	5
PR12	9	2720	5	5
PR0		1394	5	5

Table 1: Distribution between the tadpole prototypes of the 203580 different integrals occurring in the calculation of the gluon propagator in linear gauge limited to at most one power of  $\xi$ . “Denominator powers” denote in fact the total number of dots on the lines.

DiaGen/IdSolver [15]. We found it also a good opportunity to study the efficiency of this approach on a large scale problem.

Since the calculation was performed in the linear gauge, the gluon propagator had the form

$$iD_{\mu\nu}(k) = \frac{i}{k^2} \left( -g_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2} \right). \quad (1)$$

It is clear that this implies that every power of the gauge parameter will lead to more powers of the denominators and irreducible numerators in the integrals. A minimal way to test gauge invariance at the end is to keep at most a single power of  $\xi$ , which corresponds to a first order expansion of the result around the Feynman gauge. Under this restriction, the calculation involved a little less than 210.000 independent integrals. The distribution of them among the four-loop tadpole prototypes in the hardest case of the gluon propagator is given in Tab. 1. The remaining cases of the quark-gluon vertex, and quark and ghost propagators involved about one-third of these integrals and one power of denominator and irreducible numerator less for each of the prototypes.

It turned out that to reduce all of the integrals to masters, it was sufficient to generate integration-by-parts identities with up to six additional powers of the denominators and four of the numerators for all of the integrals up to seven lines, and with five additional

powers of the denominators and four of the numerators for the eight- and nine-liners. The total number of solved integrals was then about 2.000.000, meaning a 10% efficiency. About 37% percent of the integrals turned out to be finite. These could have been eliminated from the very beginning by a careful study of divergences. We convinced ourselves, however, that this would not allow for lower powers of denominators and numerators in the reduction process, unless some very involved procedure were used.

### 3 Results

Since the  $\beta$ -function is, up to normalization, the anomalous dimension of the coupling constant, it is necessary to perform the complete renormalization of some vertex. To this end we chose the quark-gluon interaction, mostly because the calculation of [4] involved the ghost and gluon instead.

The renormalization constant of the quark-gluon vertex will subsequently be denoted by  $Z_1$ , whereas the renormalization constants of the quark, gluon and ghost fields by  $Z_2$ ,  $Z_3$  and  $Z_3^c$  respectively. Even though  $Z_3^c$  is not necessary for the present calculation, we derived it in order to have the complete set of renormalization constants at the four-loop level.

We will not give the renormalization constants explicitly, but instead we will limit ourselves to the anomalous dimensions, which are defined by

$$\gamma = -\mu^2 \frac{d \log Z}{d \mu^2}, \quad (2)$$

where  $\mu$  is the 't Hooft unit of mass introduced to keep the renormalized coupling constant dimensionless. Since we use dimensional regularization and the  $\overline{\text{MS}}$  scheme, the renormalization constants can be expanded as

$$Z = 1 + \sum_{i=1}^{\infty} \frac{z^{(i)}(a_s, \xi)}{\epsilon^i} = 1 + \sum_{i=1}^{\infty} \sum_{j=1}^i a_s^i \frac{z^{(i,j)}(\xi)}{\epsilon^j}, \quad (3)$$

where  $a_s$  is connected to the QCD coupling constant  $g$  by  $a_s = \alpha_s/(4\pi) = g^2/(16\pi^2)$ . Using the fact that the dependence of  $Z$  on  $\mu$  enters only through  $a_s$  and  $\xi$ , and that the renormalization constants of the gluon field and of the gauge parameter are equal, one obtains

$$(-\epsilon + \beta)a_s \frac{\partial \log Z}{\partial a_s} - \gamma_3(1 - \xi) \frac{\partial \log Z}{\partial \xi} = -\gamma, \quad (4)$$

where the  $\beta$ -function is simply equal to the anomalous dimension of  $\alpha_s$ , *i.e.*  $\beta = \gamma_{\alpha_s}$ . This implies that

$$\gamma = a_s \frac{\partial z^{(1)}}{\partial a_s} = - \sum_{i=0}^{\infty} a_s^{i+1} \gamma^{(i)}, \quad (5)$$

where now

$$\gamma^{(i)} = -(i+1)z^{(i+1,1)}. \quad (6)$$

Eq. 4 can be used in turn to reconstruct the original renormalization constant from a given anomalous dimension.

Since the quark-gluon vertex, as well as the quark and gluon anomalous dimensions have already been given up to the three-loop level in the linear gauge in [3], we will not reproduce them here<sup>2</sup> but only give our result for the four-loop anomalous dimensions with color structures of a general compact simple Lie group. At this point we stress once more, that the results have been obtained in a first order expansion around the Feynman gauge

$$\begin{aligned}
\gamma_1^{(3)} = & C_A C_F T_F^2 n_f^2 \left( \frac{7870}{243} - \frac{8}{3} \zeta_3 + 24 \zeta_4 \right) + C_A C_F^2 T_F n_f \left( -\frac{797}{18} + 118 \zeta_3 + 36 \zeta_4 + 40 \zeta_5 \right) \\
& + C_A C_F^3 \left( \frac{5131}{12} + 848 \zeta_3 - 1440 \zeta_5 \right) + C_A T_F^3 n_f^3 \left( -\frac{166}{81} + \frac{32}{9} \zeta_3 \right) \\
& + C_A^2 C_F T_F n_f \left( -\frac{104542}{243} + \frac{187}{3} \zeta_3 - 88 \zeta_4 - 20 \zeta_5 \right) + C_F^2 T_F^2 n_f^2 \left( \frac{304}{9} - 32 \zeta_3 \right) \\
& + C_A^2 C_F^2 \left( -\frac{23777}{36} - 214 \zeta_3 - 66 \zeta_4 + 790 \zeta_5 \right) + C_A^2 T_F^2 n_f^2 \left( \frac{6307}{972} + \frac{94}{3} \zeta_3 - 18 \zeta_4 \right) \\
& + C_A^3 C_F \left( \frac{10059589}{15552} - \frac{1489}{24} \zeta_3 + \frac{173}{4} \zeta_4 - \frac{1865}{12} \zeta_5 \right) + \frac{280}{81} C_F T_F^3 n_f^3 \\
& + C_A^3 T_F n_f \left( -\frac{473903}{7776} - \frac{3311}{24} \zeta_3 + \frac{387}{8} \zeta_4 + 55 \zeta_5 \right) + C_F^3 T_F n_f \left( \frac{76}{3} - 64 \zeta_3 \right) \\
& + C_A^4 \left( \frac{350227}{3888} + \frac{2959}{72} \zeta_3 - \frac{111}{32} \zeta_4 - \frac{5125}{96} \zeta_5 \right) + C_F^4 \left( -\frac{1027}{8} - 400 \zeta_3 + 640 \zeta_5 \right) \\
& + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} (-48 \zeta_3 + 60 \zeta_5) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( -\frac{21}{8} + \frac{367}{4} \zeta_3 - \frac{335}{4} \zeta_5 \right) \\
& + 128 n_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} + \frac{d_F^{abcd} d_A^{abcd}}{N_F} (-66 + 190 \zeta_3 - 170 \zeta_5) \\
& + \xi \left( C_A C_F T_F^2 n_f^2 \left( \frac{1076}{243} - \frac{16}{3} \zeta_3 \right) + C_A C_F^2 T_F n_f \left( \frac{767}{12} - 44 \zeta_3 - 12 \zeta_4 \right) \right. \\
& + C_A^2 C_F T_F n_f \left( \frac{7423}{243} + \frac{76}{3} \zeta_3 + \frac{9}{2} \zeta_4 \right) + C_A^2 C_F^2 \left( -3 + \frac{7}{2} \zeta_3 - 5 \zeta_5 \right) \\
& + C_A^2 T_F^2 n_f^2 \left( \frac{1229}{972} - \frac{4}{3} \zeta_3 \right) + C_A^3 C_F \left( -\frac{2127929}{31104} - \frac{1013}{24} \zeta_3 + \frac{87}{16} \zeta_4 + \frac{665}{24} \zeta_5 \right) \\
& + C_A^3 T_F n_f \left( \frac{35345}{7776} + \frac{37}{3} \zeta_3 + \frac{13}{8} \zeta_4 \right) + C_A^4 \left( -\frac{1539403}{62208} - \frac{389}{32} \zeta_3 + \frac{73}{32} \zeta_4 + \frac{55}{8} \zeta_5 \right) \\
& \left. + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( \frac{9}{16} - \frac{139}{8} \zeta_3 \right) + \frac{d_F^{abcd} d_A^{abcd}}{N_F} (-1 - 48 \zeta_3 + 70 \zeta_5) \right), \quad (7)
\end{aligned}$$

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<sup>2</sup>During the course of our calculation we found full agreement with [3]. Contrary to the four-loop case, our three-loop calculation was performed without expansion in the gauge parameter.

$$\begin{aligned}
\gamma_2^{(3)} = & C_A C_F T_F^2 n_f^2 \left( \frac{6835}{243} + \frac{112}{3} \zeta_3 \right) + C_A C_F^2 T_F n_f \left( -\frac{2407}{36} + 44\zeta_3 + 36\zeta_4 + 160\zeta_5 \right) \\
& + C_A C_F^3 \left( \frac{5131}{12} + 848\zeta_3 - 1440\zeta_5 \right) + C_A^2 C_F^2 \left( -\frac{23777}{36} - 214\zeta_3 - 66\zeta_4 + 790\zeta_5 \right) \\
& + C_A^2 C_F T_F n_f \left( -\frac{1365691}{3888} - \frac{119}{3} \zeta_3 - 25\zeta_4 - 80\zeta_5 \right) + \frac{280}{81} C_F T_F^3 n_f^3 \\
& + C_A^3 C_F \left( \frac{10059589}{15552} - \frac{1489}{24} \zeta_3 + \frac{173}{4} \zeta_4 - \frac{1865}{12} \zeta_5 \right) + C_F^2 T_F^2 n_f^2 \left( \frac{304}{9} - 32\zeta_3 \right) \\
& + C_F^3 T_F n_f \left( \frac{76}{3} - 64\zeta_3 \right) + C_F^4 \left( -\frac{1027}{8} - 400\zeta_3 + 640\zeta_5 \right) + 128n_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} \\
& + \frac{d_F^{abcd} d_A^{abcd}}{N_F} (-66 + 190\zeta_3 - 170\zeta_5) \\
+ \xi & \left( C_A C_F T_F^2 n_f^2 \left( \frac{1076}{243} - \frac{16}{3} \zeta_3 \right) + C_A C_F^2 T_F n_f \left( \frac{767}{12} - 44\zeta_3 - 12\zeta_4 \right) \right. \\
& + C_A^2 C_F T_F n_f \left( \frac{48865}{3888} + \frac{118}{3} \zeta_3 + \frac{15}{2} \zeta_4 \right) + C_A^2 C_F^2 \left( -3 + \frac{7}{2} \zeta_3 - 5\zeta_5 \right) \\
& + C_A^3 C_F \left( -\frac{2127929}{31104} - \frac{1013}{24} \zeta_3 + \frac{87}{16} \zeta_4 + \frac{665}{24} \zeta_5 \right) \\
& \left. + \frac{d_F^{abcd} d_A^{abcd}}{N_F} (-1 - 48\zeta_3 + 70\zeta_5) \right), \tag{8}
\end{aligned}$$

$$\begin{aligned}
\gamma_3^{(3)} = & C_A C_F T_F^2 n_f^2 \left( -\frac{15082}{243} - \frac{1168}{9} \zeta_3 + 48\zeta_4 \right) + C_A C_F^2 T_F n_f \left( \frac{10847}{54} + \frac{980}{9} \zeta_3 - 240\zeta_5 \right) \\
& + C_A T_F^3 n_f^3 \left( -\frac{1420}{243} + \frac{64}{9} \zeta_3 \right) + C_A^2 C_F T_F n_f \left( -\frac{363565}{1944} + \frac{2492}{9} \zeta_3 - 126\zeta_4 + 120\zeta_5 \right) \\
& + C_A^2 T_F^2 n_f^2 \left( -\frac{41273}{486} + \frac{340}{9} \zeta_3 - 36\zeta_4 \right) + C_F^2 T_F^2 n_f^2 \left( -\frac{1352}{27} + \frac{704}{9} \zeta_3 \right) \\
& + C_A^3 T_F n_f \left( \frac{1404961}{3888} - \frac{1285}{4} \zeta_3 + \frac{387}{4} \zeta_4 + 110\zeta_5 \right) - 46C_F^3 T_F n_f \\
& + C_A^4 \left( -\frac{252385}{1944} + \frac{1045}{12} \zeta_3 - \frac{111}{16} \zeta_4 - \frac{5125}{48} \zeta_5 \right) - \frac{1232}{243} C_F T_F^3 n_f^3 \\
& + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \left( -\frac{512}{9} + \frac{1376}{3} \zeta_3 + 120\zeta_5 \right) + n_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \left( \frac{704}{9} - \frac{512}{3} \zeta_3 \right) \\
& + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( \frac{131}{36} - \frac{307}{6} \zeta_3 - \frac{335}{2} \zeta_5 \right) \\
+ \xi & \left( C_A^2 C_F T_F n_f \left( \frac{863}{24} - 28\zeta_3 - 6\zeta_4 \right) + C_A^2 T_F^2 n_f^2 \left( \frac{1229}{486} - \frac{8}{3} \zeta_3 \right) \right. \\
& + C_A^3 T_F n_f \left( \frac{35345}{3888} + \frac{74}{3} \zeta_3 + \frac{13}{4} \zeta_4 \right) + C_A^4 \left( -\frac{1539403}{31104} - \frac{389}{16} \zeta_3 + \frac{73}{16} \zeta_4 + \frac{55}{4} \zeta_5 \right) \\
& \left. + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( \frac{9}{8} - \frac{139}{4} \zeta_3 \right) \right). \tag{9}
\end{aligned}$$

The first three terms of the expansion of the anomalous dimension of the ghost field in the linear gauge can be found in [5]. Even though the results there are restricted to the SU(N) group, the values for a general compact simple Lie group can be derived on the basis of the fact that only quadratic Casimir operators occur.

As before our four-loop result is obtained in a first order expansion around the Feynman gauge

$$\begin{aligned}
\gamma_3^{c(3)} = & C_A C_F T_F^2 n_f^2 \left( -\frac{115}{27} + 40\zeta_3 - 24\zeta_4 \right) + C_A C_F^2 T_F n_f \left( -\frac{271}{12} - 74\zeta_3 + 120\zeta_5 \right) \\
& + C_A T_F^3 n_f^3 \left( \frac{166}{81} - \frac{32}{9}\zeta_3 \right) + C_A^2 T_F^2 n_f^2 \left( -\frac{8315}{972} - \frac{86}{3}\zeta_3 + 18\zeta_4 \right) \\
& + C_A^2 C_F T_F n_f \left( \frac{22517}{432} - 86\zeta_3 + 69\zeta_4 - 60\zeta_5 \right) \\
& + C_A^3 T_F n_f \left( \frac{449239}{7776} + \frac{2983}{24}\zeta_3 - \frac{423}{8}\zeta_4 - 55\zeta_5 \right) \\
& + C_A^4 \left( -\frac{256337}{3888} - \frac{2485}{72}\zeta_3 + \frac{123}{32}\zeta_4 + \frac{4505}{96}\zeta_5 \right) \\
& + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} (48\zeta_3 - 60\zeta_5) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( \frac{21}{8} - \frac{299}{4}\zeta_3 + \frac{265}{4}\zeta_5 \right) \\
+ \xi & \left( C_A^2 C_F T_F n_f \left( \frac{425}{48} - 2\zeta_3 - 3\zeta_4 \right) + C_A^2 T_F^2 n_f^2 \left( \frac{779}{972} - \frac{4}{3}\zeta_3 \right) \right. \\
& + C_A^3 T_F n_f \left( -\frac{2527}{7776} + \frac{7}{4}\zeta_3 + \frac{23}{8}\zeta_4 \right) + C_A^4 \left( -\frac{256273}{62208} + \frac{199}{48}\zeta_3 - \frac{47}{16}\zeta_4 + \frac{25}{48}\zeta_5 \right) \\
& \left. + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( -\frac{9}{16} - \frac{27}{8}\zeta_3 + \frac{85}{4}\zeta_5 \right) \right). \tag{10}
\end{aligned}$$

The notation is similar to the one used in [4]. Besides Riemann  $\zeta$  functions, the result contains the quadratic Casimir operators of the fundamental and adjoint representations,  $C_F$  and  $C_A$ , as well as the normalization of the trace of the fundamental representation  $T_F \delta^{ab} = \text{Tr}(T^a T^b)$ , where  $T^a$  are the representation generators, and the number of fermion families  $n_f$ . The higher order invariants are constructed from the symmetric tensors

$$\begin{aligned}
d_F^{abcd} = & \frac{1}{6} \text{Tr} \left( T^a T^b T^c T^d + T^a T^b T^d T^c + T^a T^c T^b T^d \right. \\
& \left. + T^a T^c T^d T^b + T^a T^d T^b T^c + T^a T^d T^c T^b \right), \tag{11}
\end{aligned}$$

(and similarly for the adjoint representation) and of the dimensions of the fundamental and adjoint representations,  $N_F$  and  $N_A$  respectively. Using the specific values for the SU(N) group

$$T_F = \frac{1}{2}, \quad C_F = \frac{N^2 - 1}{2N}, \quad C_A = N, \quad \frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{N^4 - 6N^2 + 18}{96N^2}, \tag{12}$$

$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N(N^2 + 6)}{48}, \quad \frac{d_A^{abcd} d_A^{abcd}}{N_A} = \frac{N^2(N^2 + 36)}{24}, \quad N_A = N^2 - 1, \quad (13)$$

we checked that Eqs. 7-10 are in perfect agreement with [5].

We can now combine the anomalous dimensions to reach the goal of our calculation, *i.e.* the four-loop  $\beta$ -function. Since,  $Z_{\alpha_s} = Z_g^2 = Z_1^2 Z_2^{-2} Z_3^{-1}$ , we have  $\beta = 2\gamma_1 - 2\gamma_2 - \gamma_3$ , and

$$\begin{aligned} \beta_3 &= C_A C_F T_F^2 n_f^2 \left( \frac{17152}{243} + \frac{448}{9} \zeta_3 \right) + C_A C_F^2 T_F n_f \left( -\frac{4204}{27} + \frac{352}{9} \zeta_3 \right) + \frac{424}{243} C_A T_F^3 n_f^3 \\ &\quad + C_A^2 C_F T_F n_f \left( \frac{7073}{243} - \frac{656}{9} \zeta_3 \right) + C_A^2 T_F^2 n_f^2 \left( \frac{7930}{81} + \frac{224}{9} \zeta_3 \right) + \frac{1232}{243} C_F T_F^3 n_f^3 \\ &\quad + C_A^3 T_F n_f \left( -\frac{39143}{81} + \frac{136}{3} \zeta_3 \right) + C_A^4 \left( \frac{150653}{486} - \frac{44}{9} \zeta_3 \right) + C_F^2 T_F^2 n_f^2 \left( \frac{1352}{27} - \frac{704}{9} \zeta_3 \right) \\ &\quad + 46 C_F^3 T_F n_f + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \left( \frac{512}{9} - \frac{1664}{3} \zeta_3 \right) + n_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \left( -\frac{704}{9} + \frac{512}{3} \zeta_3 \right) \\ &\quad + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( -\frac{80}{9} + \frac{704}{3} \zeta_3 \right). \end{aligned} \quad (14)$$

This result, manifestly gauge invariant, confirms [4]. For completeness, we reproduce the lower order values, which we have also calculated

$$\begin{aligned} \beta_2 &= \frac{2857}{54} C_A^3 - \frac{1415}{27} C_A^2 T_F n_f + \frac{158}{27} C_A T_F^2 n_f^2 + \frac{44}{9} C_F T_F^2 n_f^2 \\ &\quad - \frac{205}{9} C_F C_A T_F n_f + 2 C_F^2 T_F n_f, \\ \beta_1 &= \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F n_f - 4 C_F T_F n_f, \\ \beta_0 &= \frac{11}{3} C_A - \frac{4}{3} T_F n_f. \end{aligned} \quad (15)$$

## 4 Conclusions

We have completed the four-loop  $\overline{\text{MS}}$  renormalization program of an unbroken gauge theory with fermions and a general compact simple Lie group in the linear gauge. The fact that we performed an expansion in the gauge parameter still allows for gauge invariance tests in practical calculations, although special gauges such as the Landau gauge cannot be chosen. If such need would occur, one may use the SU(N) results from [5]. The correctness of the latter relies, however, on the four-loop  $\beta$ -function. Therefore, the present work was not only indispensable to confirm the value of the four-loop  $\beta$ -function itself, but also the correctness of [5]. This goal has been reached.

It should be stressed that with the accumulated solved integrals, one could easily obtain one more term in the  $\xi$  expansion of the anomalous dimensions. This is due to the fact that the complexity of the integrals occurring in the gluon propagator with at most a single power of  $\xi$  is comparable to the complexity of the integrals for the other functions with

at most two powers of  $\xi$ . Having solved the latter, the gauge invariant  $\beta$ -function enables then to recover the third term in the  $\xi$  expansion of the gluon propagator as well. We did not perform such a calculation, since from the purely pragmatical point of view, only the complete  $\xi$  dependence would be an improvement.

A second comment concerns the independence of our calculation. The reader should notice that besides basic software as FORM [9], Fermat [16] *etc.* and our own programs [15], the only external package that we used was the Color package [17] of the FORM distribution. On a diagram per diagram basis, we made sure that the color factors produced by this package agree with the SU(N) values obtained with the algorithm of [18]. Our confidence was strengthened by the fact that we agreed with [5] on the final results.

## 5 Acknowledgements

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## A Master integral expansions

Here we give the  $\epsilon$  expansions of the master integrals depicted in Fig. 1 up to the necessary order. The notation is similar to that used in MATAD [19], *i.e.* we use the integration measure

$$(e^{\epsilon\gamma_E})^4 \int \prod_{i=1}^4 \frac{d^d k_i}{i\pi^{d/2}}, \quad (16)$$

with  $d = 4 - 2\epsilon$  and the constants which occur in the divergent parts of the integrals are

$$\begin{aligned} S2 &= \frac{4}{9\sqrt{3}} \text{Cl}_2\left(\frac{\pi}{3}\right), \\ T1ep &= -\frac{45}{2} - \frac{\pi\sqrt{3}\log^2 3}{8} - \frac{35\pi^3\sqrt{3}}{216} - \frac{9}{2}\zeta_2 + \zeta_3 \\ &\quad + 6\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right) - 6\sqrt{3}\text{Im}\left(\text{Li}_3\left(\frac{e^{-i\frac{\pi}{6}}}{\sqrt{3}}\right)\right), \\ D6 &= 6\zeta_3 - 17\zeta_4 - 4\zeta_2\log^2 2 + \frac{2}{3}\log^4 2 + 16\text{Li}_4\left(\frac{1}{2}\right) - 4\left(\text{Cl}_2\left(\frac{\pi}{3}\right)\right)^2, \end{aligned} \quad (17)$$

where  $\text{Cl}_2(x) = \text{Im}(\text{Li}_2(e^{ix}))$  is the Clausen function. Notice, however, that we use the Minkowski space metric and “Minkowski type” propagators, *i.e.*  $1/(k^2 - m^2)$ .

$$\mathbf{PR0} = \mathcal{O}(\epsilon^0),$$

$$\mathbf{PR12} = \mathcal{O}(\epsilon^0),$$

$$\mathbf{PR15} = \frac{1}{\epsilon^2} \frac{3}{2} \zeta_3 + \frac{1}{\epsilon} \left( D6 + \frac{3}{2} \zeta_3 - \frac{3}{4} \zeta_4 \right) + \mathcal{O}(\epsilon^0),$$

$$\mathbf{PR11} = \frac{5}{\epsilon} \zeta_5 + \mathcal{O}(\epsilon^0),$$

$$\mathbf{PR11d} = \mathcal{O}(\epsilon^0),$$

$$\begin{aligned} \mathbf{PR14} = & \frac{1}{\epsilon^4} \frac{3}{4} + \frac{1}{\epsilon^3} \frac{25}{4} + \frac{1}{\epsilon^2} \left( \frac{137}{4} - \frac{81}{2} S2 + \frac{3}{2} \zeta_2 \right) \\ & + \frac{1}{\epsilon} \left( \frac{363}{4} - 162 S2 - 3 T1ep - \zeta_2 - \frac{27}{2} \zeta_3 \right) + \mathcal{O}(\epsilon^0), \end{aligned}$$

$$\mathbf{PR13} = \frac{2}{\epsilon^2} \zeta_3 + \frac{1}{\epsilon} (D6 + 2\zeta_3) + \mathcal{O}(\epsilon^0),$$

$$\begin{aligned} \mathbf{PR10} = & \frac{1}{\epsilon^4} \frac{1}{2} + \frac{1}{\epsilon^3} \frac{53}{12} + \frac{1}{\epsilon^2} \left( \frac{51}{2} - 27 S2 + \zeta_2 \right) \\ & + \frac{1}{\epsilon} \left( \frac{937}{12} - \frac{243}{2} S2 - 2 T1ep - \frac{1}{6} \zeta_2 - \frac{32}{3} \zeta_3 \right) + \mathcal{O}(\epsilon^0), \end{aligned}$$

$$\begin{aligned} \mathbf{PR9} = & \frac{1}{\epsilon^4} \frac{1}{4} + \frac{1}{\epsilon^3} \frac{7}{3} + \frac{1}{\epsilon^2} \left( \frac{169}{12} - \frac{27}{2} S2 + \frac{1}{2} \zeta_2 + \zeta_3 \right) \\ & + \frac{1}{\epsilon} \left( \frac{143}{3} - \frac{135}{2} S2 - T1ep + \frac{1}{6} \zeta_2 - \frac{4}{3} \zeta_3 + \frac{3}{2} \zeta_4 \right) + \mathcal{O}(\epsilon^0), \end{aligned}$$

$$\begin{aligned} \mathbf{PR9d} = & \frac{1}{\epsilon^4} \frac{1}{12} + \frac{1}{\epsilon^3} \frac{1}{3} + \frac{1}{\epsilon^2} \left( \frac{7}{12} - \frac{9}{2} S2 + \frac{1}{6} \zeta_2 \right) \\ & + \frac{1}{\epsilon} \left( -\frac{26}{3} + \frac{27}{2} S2 - \frac{1}{3} T1ep - \frac{5}{6} \zeta_2 + \frac{29}{9} \zeta_3 \right) + \mathcal{O}(\epsilon^0), \end{aligned}$$

$$\begin{aligned} \mathbf{PR8} = & \frac{1}{\epsilon^4} \frac{9}{4} + \frac{1}{\epsilon^3} \frac{27}{2} + \frac{1}{\epsilon^2} \left( \frac{207}{4} - \frac{81}{2} S2 + \frac{9}{2} \zeta_2 \right) \\ & + \frac{1}{\epsilon} \left( \frac{189}{2} - \frac{243}{2} S2 - 3 T1ep + \frac{27}{2} \zeta_2 \right) + \mathcal{O}(\epsilon^0), \end{aligned}$$

$$\begin{aligned} \mathbf{PR7} = & \frac{1}{\epsilon^4} \frac{13}{8} + \frac{1}{\epsilon^3} \frac{491}{48} + \frac{1}{\epsilon^2} \left( \frac{3719}{96} - \frac{81}{4} S2 + \frac{13}{4} \zeta_2 \right) \\ & + \frac{1}{\epsilon} \left( \frac{13741}{192} - \frac{459}{8} S2 - \frac{3}{2} T1ep + \frac{329}{24} \zeta_2 - \frac{2}{3} \zeta_3 \right) \\ & + \frac{381313}{10368} - \frac{1593}{16} S2 - \frac{17}{4} T1ep - \frac{3}{2} T1ep2 - \frac{5}{4} PR4dfn \\ & - \frac{1}{4} PR4fin + \frac{6805}{144} \zeta_2 - \frac{81}{4} S2 \zeta_2 + \frac{61}{2} \zeta_3 + \frac{153}{16} \zeta_4 + \mathcal{O}(\epsilon), \end{aligned}$$

$$\begin{aligned} \mathbf{PR6} = & \frac{1}{\epsilon^4} + \frac{1}{\epsilon^3} \frac{20}{3} + \frac{1}{\epsilon^2} (29 - 27 S2 + 2 \zeta_2) \\ & + \frac{1}{\epsilon} \left( \frac{181}{3} - 81 S2 - 2 T1ep + \frac{13}{3} \zeta_2 - \frac{16}{3} \zeta_3 \right) + \mathcal{O}(\epsilon^0), \end{aligned}$$

$$\mathbf{PR5} = \frac{1}{\epsilon^4} \frac{3}{2} + \frac{1}{\epsilon^3} \frac{19}{2} + \frac{1}{\epsilon^2} \left( \frac{67}{2} + 3 \zeta_2 \right) + \frac{1}{\epsilon} \left( \frac{127}{2} + 19 \zeta_2 - 5 \zeta_3 \right) + \mathcal{O}(\epsilon^0),$$

$$\begin{aligned} \mathbf{PR4} = & \frac{1}{\epsilon^4} \frac{5}{2} + \frac{1}{\epsilon^3} \frac{35}{3} + \frac{1}{\epsilon^2} \left( \frac{4565}{144} + 5 \zeta_2 \right) \\ & + \frac{1}{\epsilon} \left( \frac{58345}{864} + \frac{70}{3} \zeta_2 - \frac{10}{3} \zeta_3 \right) + PR4fin + \mathcal{O}(\epsilon), \end{aligned}$$

$$\mathbf{PR4d} = -\frac{1}{\epsilon^3} \frac{7}{6} - \frac{1}{\epsilon^2} \frac{215}{48} + \frac{1}{\epsilon} \left( -\frac{965}{96} - \frac{7}{3} \zeta_2 \right) + PR4dfn + \mathcal{O}(\epsilon),$$

$$\begin{aligned} \mathbf{PR3} = & \frac{1}{\epsilon^4} \frac{3}{2} + \frac{1}{\epsilon^3} \frac{15}{2} + \frac{1}{\epsilon^2} \left( 24 - \frac{27}{2} S2 + 3 \zeta_2 \right) \\ & + \frac{1}{\epsilon} \left( \frac{81}{2} - 27 S2 - T1ep + \frac{21}{2} \zeta_2 - \zeta_3 \right) + 57 - \frac{81}{2} S2 - 2 T1ep \\ & - T1ep2 + \frac{57}{2} \zeta_2 - \frac{27}{2} S2 \zeta_2 - 5 \zeta_3 + \frac{51}{8} \zeta_4 + \mathcal{O}(\epsilon), \end{aligned}$$

$$\begin{aligned} \mathbf{PR2} = & \frac{2}{\epsilon^4} + \frac{1}{\epsilon^3} \frac{29}{3} + \frac{1}{\epsilon^2} \left( \frac{163}{6} + 4 \zeta_2 \right) + \frac{1}{\epsilon} \left( \frac{601}{12} + \frac{58}{3} \zeta_2 - \frac{8}{3} \zeta_3 \right) \\ & + \frac{635}{24} + \frac{163}{3} \zeta_2 + \frac{220}{9} \zeta_3 + 12 \zeta_4 + \mathcal{O}(\epsilon), \end{aligned}$$

$$\begin{aligned} \mathbf{PR1} = & \frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \frac{1}{\epsilon^2} (10 + 2 \zeta_2) + \frac{1}{\epsilon} \left( 20 + 8 \zeta_2 - \frac{4}{3} \zeta_3 \right) \\ & + 35 + 20 \zeta_2 - \frac{16}{3} \zeta_3 + 6 \zeta_4 + \mathcal{O}(\epsilon). \end{aligned}$$

The expansions of the irreducible four-loop integrals have been obtained either from the finiteness of the same integral with higher powers of denominators or with the help of the method described in [12]. After taking into account the different normalization and translating master integrals with dots to integrals with suitable irreducible numerators, the above results for the divergent parts are in agreement with the numerical values obtained in [6].

The integrals PR1-PR4, PR4d and PR7 are needed up to constant parts. It is, however, sufficient to express the finite part of PR7 through the finite parts of PR4 and PR4d and keep them as symbols, since they always cancel from the divergent part of any four-loop vacuum integral.

The reader should notice that we also needed  $\epsilon$  expansions of two- and three-loop tadpoles. These have been obtained in [20, 21]. The values can be read off from our results on the reducible four-loop integrals above.

## References

- [1] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343;  
H. D. Politzer, Phys. Rev. Lett. **30** (1973) 1346;  
G. 't Hooft, report at the Marseille Conference on Yang-Mills Fields, 1972.
- [2] W. E. Caswell, Phys. Rev. Lett. **33** (1974) 244;  
D. R. T. Jones, Nucl. Phys. B **75** (1974) 531;  
E. Egorian and O. V. Tarasov, Theor. Math. Phys. **41** (1979) 863 [Teor. Mat. Fiz. **41** (1979) 26].
- [3] O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B **93** (1980) 429;  
S. A. Larin and J. A. M. Vermaasen, Phys. Lett. B **303** (1993) 334.
- [4] T. van Ritbergen, J. A. M. Vermaasen and S. A. Larin, Phys. Lett. B **400** (1997) 379.

- [5] K. G. Chetyrkin, arXiv:hep-ph/0405193.
- [6] S. Laporta, Phys. Lett. B **549**, 115 (2002).
- [7] A. A. Vladimirov, Theor. Math. Phys. **43** (1980) 417 [Teor. Mat. Fiz. **43** (1980) 210].
- [8] K. G. Chetyrkin and V. A. Smirnov, Phys. Lett. B **144** (1984) 419;  
 K. G. Chetyrkin, Phys. Lett. B **390** (1997) 309;  
 K. G. Chetyrkin, Phys. Lett. B **391** (1997) 402.
- [9] J. A. M. Vermaseren, arXiv:math-ph/0010025.
- [10] S. A. Larin, F. V. Tkachov and J. A. M. Vermaseren, NIKHEF-H-91-18.
- [11] M. Misiak and M. Munz, Phys. Lett. B **344** (1995) 308.
- [12] K. G. Chetyrkin, M. Misiak and M. Munz, Nucl. Phys. B **518** (1998) 473.
- [13] Y. Schröder, Nucl. Phys. Proc. Suppl. **116** (2003) 402.
- [14] S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087.
- [15] M. Czakon, DiaGen/IdSolver, *unpublished*.
- [16] R. H. Lewis, Fermat, <http://www.bway.net/~lewis/>.
- [17] T. van Ritbergen, A. N. Schellekens and J. A. M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 41.
- [18] P. Cvitanovic, Phys. Rev. D **14** (1976) 1536.
- [19] M. Steinhauser, Comput. Phys. Commun. **134** (2001) 335.
- [20] A. I. Davydychev and J. B. Tausk, Nucl. Phys. B **397** (1993) 123;  
 A. I. Davydychev and J. B. Tausk, Phys. Rev. D **53** (1996) 7381;  
 A. I. Davydychev, Phys. Rev. D **61** (2000) 087701.
- [21] D. J. Broadhurst, Z. Phys. C **54** (1992) 599;  
 D. J. Broadhurst, Eur. Phys. J. C **8** (1999) 311;  
 J. Fleischer and M. Y. Kalmykov, Phys. Lett. B **470** (1999) 168;  
 K. G. Chetyrkin and M. Steinhauser, Nucl. Phys. B **573** (2000) 617.